Carrier-Sideband USO noise cancellation

Daniel Shaddock

September 25, 2002

This document outlines a scheme to cancel laser phase noise and USO noise in the final LISA phase readout. The notation of Tinto et. al. (Phys. Rev. D 2002) is adopted where subscripts are used to indicate the arm, the spacecraft and delay pertinent to a measurement or variable.

We assume that the laser phase is given by $\nu_i t + p_i(t)$. The frequency, ν_i is defined to be constant in time and all the fluctuations are considered as fluctuations in phase and are contained in $p_i(t)$. Likewise the ultra-stable oscillators (USO) have a phase of $f_i t + q_i(t)$, where the frequency is constant and fluctuations are contained in the phase.

The interferometer topology is as follows. Each spacecraft consists of two lasers[1], referred to as the carrier and subcarrier, that are phase locked with a frequency (and phase) offset determined by the frequency (and phase) of the local USO. Both lasers are sent to both of the distant spacecraft where they are interfered with the two local lasers on those spacecraft after reflecting off the proof mass. The recorded signals measure the interference between the local carrier-distant subcarrier, denoted s_{ij} and the local subcarrier-distant carrier, denoted s'_{ij}, giving two data streams per optical bench. The outputs of the phasemeters at various spacecraft are given by,

$$\begin{aligned}
s_{21} &= ((\nu_3 + f_3)(1 - \dot{L}_2) - \nu_1 - a_{21} f_1)t + p_{3,2} + q_{3,2} - p_1 - a_{21} q_1 + g w_{21} + n_{21} && (1)\\
s_{31} &= ((\nu_2 + f_2)(1 - \dot{L}_3) - \nu_1 - a_{31} f_1)t + p_{2,3} + q_{2,3} - p_1 - a_{31} q_1 + g w_{31} + n_{31} && (2)\\
s_{23} &= ((\nu_1 + f_1)(1 - \dot{L}_2) - \nu_3 - a_{23} f_3)t + p_{1,2} + q_{1,2} - p_3 - a_{23} q_3 + g w_{23} + n_{23} && (3)\\
s_{32} &= ((\nu_1 + f_1)(1 - \dot{L}_3) - \nu_2 - a_{32} f_2)t + p_{1,3} + q_{1,3} - p_2 - a_{32} q_2 + g w_{32} + n_{32} && (4)
\end{aligned}$$

$$\begin{aligned}
s'_{21} &= (\nu_3(1 - \dot{L}_2) - (\nu_1 + f_1) - a'_{21} f_1)t + p_{3,2} - p_1 - q_1 - a'_{21} q_1 + g w_{21} + n'_{21} && (5)\\
s'_{31} &= (\nu_2(1 - \dot{L}_3) - (\nu_1 + f_1) - a'_{31} f_1)t + p_{2,3} - p_1 - q_1 - a'_{31} q_1 + g w_{31} + n'_{31} && (6)\\
s'_{23} &= (\nu_1(1 - \dot{L}_2) - (\nu_3 + f_3) - a'_{23} f_3)t + p_{1,2} - p_3 - q_3 - a'_{23} q_3 + g w_{23} + n'_{23} && (7)\\
s'_{32} &= (\nu_1(1 - \dot{L}_3) - (\nu_2 + f_2) - a'_{32} f_2)t + p_{1,3} - p_2 - q_2 - a'_{32} q_2 + g w_{32} + n'_{32} && (8)
\end{aligned}$$

[1]In reality there will be four lasers per spacecraft (2 per bench) however in the absence of bench noise, or with phase locking this need not be included. For simplicity we assume that the lasers on adjacent benches are identical in this calculation.

1 Phase locking

We will phase lock the lasers at the far spacecraft (2 and 3) to the lasers in spacecraft 1. The result of the locking will be that the carriers have the same frequencies and the sidebands have the same frequencies (as measured at the distant spacecraft). To achieve this we will impose the following conditions (in the limit of high gain):

$$\begin{aligned} s_{23} &\rightarrow 0 \quad \text{via feedback to } \nu_3,\ p_3 && (9)\\ s_{32} &\rightarrow 0 \quad \text{via feedback to } \nu_2,\ p_2 && (10)\\ s'_{23} &\rightarrow 0 \quad \text{via feedback to } f_3,\ q_3 && (11)\\ s'_{32} &\rightarrow 0 \quad \text{via feedback to } f_2,\ q_2 && (12)\\ & && (13)\end{aligned}$$

We are free to choose the offset locking frequencies to be f_3, f_2, $-f_3$ and $-f_2$ respectively by setting $a_{32} = a_{23} = 1$ and $a'_{32} = a'_{23} = -1$. These conditions impose the following equalities on the frequency and phase of the lasers and USO's on spacecraft 2 and 3.

$$\nu_3 = \nu_1(1 - \dot{L}_2), \qquad p_3 = p_{1,2} + gw_{23} + n'_{23} \tag{14}$$

$$\nu_2 = \nu_1(1 - \dot{L}_3), \qquad p_2 = p_{1,3} + gw_{32} + n'_{32} \tag{15}$$

$$f_3 = f_1(1 - \dot{L}_2), \qquad q_3 = q_{1,2} + n_{23} - n'_{23} \tag{16}$$

$$f_2 = f_1(1 - \dot{L}_3), \qquad q_2 = q_{1,3} + n_{32} - n'_{32} \tag{17}$$

An important point is that the beatnotes measured on the photodetector are held at a frequency $f_2 \approx f_3 \approx f_1$ where the photocurrent is free from low frequency technical (electronic and optical) noise. Substituting these parameters into the remaining four data streams we obtain,

$$\begin{aligned} s_{21} &= (f_1(1 - 2\dot{L}_2) - 2\dot{L}_2\nu_1 - a_{21}f_1)t \\ &\quad + p_{1,22} - p_1 + q_{1,22} - a_{21}q_1 + gw_{23,2} + gw_{21} + n_{23,2} + n_{21} && (18)\\ s_{31} &= (f_1(1 - 2\dot{L}_3) - 2\dot{L}_3\nu_1 - a_{31}f_1)t \\ &\quad + p_{1,33} - p_1 + q_{1,33} - a_{31}q_1 + gw_{32,3} + gw_{31} + n_{32,3} + n_{31} && (19)\end{aligned}$$

$$\begin{aligned} s'_{21} &= (\nu_1(1 - 2\dot{L}_2) - \nu_1 - f_1 - a'_{21}f_1)t \\ &\quad + p_{1,22} - p_1 - q_1(1 + a'_{21}) + gw_{23,2} + gw_{21} + n'_{23,2} + n'_{21} && (20)\\ s'_{31} &= (\nu_1(1 - 2\dot{L}_3) - \nu_1 - f_1 - a'_{31}f_1)t \\ &\quad + p_{1,33} - p_1 - q_1(1 + a'_{31}) + gw_{32,3} + gw_{31} + n'_{32,3} + n'_{31} && (21)\end{aligned}$$

The coefficients a_{21}, a_{31}, a'_{21}, and a'_{31} must be chosen to mix the beatnote down to zero frequency. The values of these coefficients must then be,

$$a_{21} = 1 - 2\dot{L}_2 - \frac{2\dot{L}_2\nu_1}{f_1} \tag{22}$$

$$a_{31} = 1 - 2\dot{L}_3 - \frac{2\dot{L}_3\nu_1}{f_1} \tag{23}$$

$$a'_{21} = \frac{-2\dot{L}_2\nu_1}{f_1} - 1 \tag{24}$$

$$a'_{31} = \frac{-2\dot{L}_3\nu_1}{f_1} - 1 \tag{25}$$

Note these coefficients have the relationships $a'_{21} = a_{21} - 2(1 - \dot{L}_2)$ and $a'_{31} = a_{31} - 2(1 - \dot{L}_3)$. Rewriting equations 18-21 with these values for the coefficients we obtain

$$\begin{aligned} s_{21} &= p_{1,22} - p_1 + q_{1,22} - a_{21}q_1 + gw_{23,2} + gw_{21} + n_{23,2} + n_{21} && (26) \\ s_{31} &= p_{1,33} - p_1 + q_{1,33} - a_{31}q_1 + gw_{32,3} + gw_{31} + n_{32,3} + n_{31} && (27) \end{aligned}$$

$$\begin{aligned} s'_{21} &= p_{1,22} - p_1 - q_1(a_{21} - (1 - 2\dot{L}_2)) + gw_{23,2} + gw_{21} + n'_{23,2} + n'_{21} && (28) \\ s'_{31} &= p_{1,33} - p_1 - q_1(a_{31} - (1 - 2\dot{L}_3)) + gw_{32,3} + gw_{31} + n'_{32,3} + n'_{31} && (29) \end{aligned}$$

The differences between the carrier-subcarrier and subcarrier-carrier beats are equal to,

$$\begin{aligned} r_{21} \equiv s_{21} - s'_{21} &= q_{1,22} - q_1(1 - 2\dot{L}_2) + n_{23,2} - n'_{23,2} + n_{21} - n'_{21} && (30) \\ r_{31} \equiv s_{31} - s'_{31} &= q_{1,33} - q_1(1 - 2\dot{L}_3) + n_{32,3} - n'_{32,3} + n_{31} - n'_{31} && (31) \end{aligned}$$

2 Time Delay Interferometry

The time delay equations in the absence of USO noise callibration for the Michelson type signal readout, X_q is given by,

$$\begin{aligned} X_q &= s_{21} - s_{21,33} - s_{31} + s_{31,22} && (32) \\ &= p_{1,22} - p_1 + q_{1,22} - a_{21}q_1 + gw_{23,2} + gw_{21} + n_{23,2} + n_{21} \\ &\quad -p_{1,2233} + p_{1,33} - q_{1,2233} + a_{21}q_{1,33} - gw_{23,233} - gw_{21,33} - n_{23,233} - n_{21,33} \\ &\quad -p_{1,33} + p_1 - q_{1,33} + a_{31}q_1 - gw_{32,3} - gw_{31} - n_{32,3} - n_{31} \\ &\quad +p_{1,3322} - p_{1,22} + q_{1,3322} - a_{31}q_{1,22} + gw_{32,322} + gw_{31,22} + n_{32,322} + n_{31,22} && (33) \\ &= q_{1,22}(1 - a_{31}) + q_1(a_{31} - a_{21}) - q_{1,33}(1 - a_{21}) \\ &\quad +gw_{21} + gw_{23,2} - gw_{21,33} - gw_{23,233} - gw_{31} - gw_{32,3} + gw_{31,22} + gw_{32,322} \\ &\quad +n_{21} + n_{23,2} - n_{21,33} - n_{23,233} - n_{31} - n_{32,3} + n_{31,22} + n_{32,322} && (34) \end{aligned}$$

In equation 34 the USO noise still enters as a noise source in this combination.

3 Ultra-stable Oscillator Noise

If we have a USO with a fractional frequency fluctuation noise spectral density of say $\Delta f_{USO}/f_{USO} \approx 10^{-13}\ /\sqrt{\mathrm{Hz}}$ then with $f_1 = 30$ MHz we will have frequency noise spectral density of $3\times10^{-6}\ \mathrm{Hz}/\sqrt{\mathrm{Hz}}$. At a signal frequency of 1 mHz this corresponds to USO phase noise spectral density $\tilde{q}_1 = 3 \times 10^{-3}$ cycles/$\sqrt{\mathrm{Hz}}$. This is above the requirements of a phase noise spectral density of the output of $\tilde{X} \sim 10^{-5}$ cycles/$\sqrt{\mathrm{Hz}}$ (corresponding to 10 pm with 1 μm light).

To remove the USO noise by including r_{21} and r_{31} we define a new combination X as,

$$X = X_q - r_{21}(1 - a_{31}) + r_{31}(1 - a_{21}) \qquad (35)$$

This does not change the gravitational wave coupling into this degree of freedom (the the r_{ij}'s contain no GW terms). However, it does change both the USO noise and shot noise present in the output.

$$
\begin{aligned}
X &= s_{21} - s_{21,33} - s_{31} + s_{31,22} - r_{21}(1 - a_{31}) + r_{31}(1 - a_{21}) \qquad (36)\\
&= q_1(2\dot{L}_3(1 - a_{21}) - 2\dot{L}_2(1 - a_{31})) \\
&\quad + gw_{21} + gw_{23,2} - gw_{21,33} - gw_{23,233} - gw_{31} - gw_{32,3} + gw_{32,322} + gw_{31,22} \\
&\quad (n_{21} + n_{23,2})a_{31} - n_{21,33} - n_{23,233} - (n_{31} + n_{32,3})a_{21} + n_{31,22} + n_{32,322} \\
&\quad + (n'_{21} + n'_{23,2})(1 - a_{31}) - (n'_{31} + n'_{32,3})(1 - a_{21}) \qquad (37)
\end{aligned}
$$

The coupling coefficient for the residual USO noise is $2\dot{L}_3(1-a_{21})-2\dot{L}_2(1-a_{31})$. After substituting the values of a_{21} and a_{31} from equations 22 and 23 we find that this coefficient is zero, indicating that the USO noise is completely suppressed.

$$
\begin{aligned}
X &= gw_{21} + gw_{23,2} - gw_{21,33} - gw_{23,233} - gw_{31} - gw_{32,3} + gw_{32,322} + gw_{31,22} \\
&\quad (n_{21} + n_{23,2})a_{31} - n_{21,33} - n_{23,233} - (n_{31} + n_{32,3})a_{21} + n_{31,22} + n_{32,322} \\
&\quad + (n'_{21} + n'_{23,2})(1 - a_{31}) - (n'_{31} + n'_{32,3})(1 - a_{21}) \qquad (38)
\end{aligned}
$$

Note that the shot noise transfer function has been substantially changed by this procedure. In equation 34 both the GW signal and noise were suppressed by the same amount, leaving the signal to noise ratio unchanged. In general this is not the case for equation 37. This is a very serious problem arising from the fact that the n_{ij} and n'_{ij} noise terms are uncorrelated and processed differently. I believe this may also be a flaw in existing TDI USO calibration schemes and plan to investigate further.

4 Outstanding Issues

Aside from this issue and verifying the real world performance of such a scheme there still remain several unsolved problems. One of the most obvious ones is how to separate the signals s_{ij} from s'_{ij}. Both signals appear with the same beatnote frequency and arrive from the distant spacecraft with the same polarization and are colinear.